Magnetically charged solutions via an analog of the electric-magnetic duality in (2+1)-dimensional gravity theories

Youngjai Kiem[a] and Dahl Park[b]

[a] *Department of Physics*

Sejong Univerity

Seoul 143-747, KOREA

E-mail: ykiem@phy.sejong.ac.kr

[b] *Department of Physics*

KAIST

Taejon 305-701, KOREA

E-mail: dpark@chep6.kaist.ac.kr

Abstract

We find an analog of the electric-magnetic duality, which is a Z_2 transformation between magnetic and electric sectors of the static and rotationally symmetric solutions in a class of (2+1)-dimensional Einstein-Maxwell-Dilaton gravity theories. The theories in our consideration include, in particular, one parameter class of theories continuously connecting the Banados-Teitelboim-Zanelli (BTZ) gravity and the low energy string effective theory. When there is no $U(1)$ charge, we have $O(2)$ or $O(1,1)$ symmetry, depending on a parameter that specifies each theory. Via the Z_2 transformation, we obtain exact magnetically charged solutions from the known electrically charged solutions. We explain the relationship between the Z_2 transformation and $O(2, Z)$ symmetry, and comment on the T-duality of the string theory.

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In general, we do not expect the existence of the electric-magnetic duality in a (2+1)-dimensional theory. This is based on an observation that, unlike the (3+1)-dimensional case [1], the number of independent components for the electric field in (2+1)-dimension is different from that of the magnetic field. If we restrict our attention only to static and rotationally symmetric field configurations in (2+1)-dimensional theories, however, we have the same number of the electric-magnetic field component and it is conceivable that some analog of the electric-magnetic duality may exist [2]. There are some reasons why we are interested in this issue. The T-duality of the string theory has been increasingly playing a significant role in recent developments of the string theory [3]. Since the target space effective action of the string theory contains $U(1)$ gauge fields from the open string sector, the T-duality may imply some analog of the electric-magnetic duality in a target space geometry. Conversely, the study of the electric-magnetic duality may lead to a better understanding of the T-duality in the low energy string theory. In particular, Cadoni [4] recently found an $O(2)$ symmetry in a class of (2+1)-dimensional Kaluza-Klein type theories that includes the low energy string effective theory without $U(1)$ gauge fields and the (uncharged) Banados-Teitelboim-Zanelli (BTZ) theory [5] [6]. It was further suggested there that the discrete version of the symmetry, i.e., $O(2,Z)$, may be related to the $O(2,2,Z)$ duality from the string theory description of the (2+1)-dimensional gravity theories [7]. There is also a more practical reason why we are interested in an analog of the electric-magnetic duality. While we have a number of exact electrically charged solutions in (2+1)-dimensional gravity theories, for example, as can be found in Ref. [8], magnetically charged solutions are relatively less understood. If we find an analog of the electric-magnetic duality, we can use it to find magnetically charged solutions from the known electrically charged solutions. In case of the BTZ theory, the magnetically charged solutions were obtained in [9]. As their results show, the properties of the magnetic solutions are too different from the electric solutions to immediately uncover any relations between them. However, the possibility of an analog of the electric-magnetic duality was suggested in further studies, for example, in [10] for the BTZ theory and in [2] for the case of the Einstein-Maxwell-Dilaton theory without the

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cosmological constant term.

In this note, we find an analog of the electric-magnetic duality for the static and rotationally symmetric solutions of the theories given by the (2+1)-dimensional action

$$I = \int d^3x \sqrt{g^{(3)}} (R^{(3)} - \frac{1}{2} g^{(3)\alpha\beta} \partial_\alpha f \partial_\beta f + \Lambda e^{bf} + \frac{1}{4} e^{\chi f} F^2) \tag{1}$$

where $R^{(3)}$, f, F denote the (2+1)-dimensional scalar curvature, the dilaton field and the curvature two form for a $U(1)$ gauge field, respectively. We use $(+ - -)$ signature for the (2+1)-dimensional metric $g^{(3)}_{\alpha\beta}$. We also have the cosmological constant Λ and two parameters b and χ, the specification of which gives us a particular gravity theory. For example, the choice $b = \chi = 0$ produces the BTZ theory, while the choice $b = -\chi = \sqrt{2}$ yields the (2+1)-dimensional low energy string effective action after a rescaling of the metric $g^{(3)}_{\alpha\beta}$. Under the assumption of the rotational symmetry, we can write the (2+1)-dimensional metric as

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta - e^{-4\phi} d\theta^2, \tag{2}$$

where the two-dimensional longitudinal metric $g_{\alpha\beta}$ and the conformal factor ϕ of the angular part of the metric is independent of the azimuthal angle θ. We choose to describe the resulting 2-dimensional longitudinal geometry of the space-time in terms of a conformal gauge, thereby setting $g_{\alpha\beta} dx^\alpha dx^\beta = - \exp(2\rho) dx^+ dx^-$. We can now reduce the action Eq. (1) to a class of 2-dimensional dilaton gravity theories [11] by integrating out the θ coordinate [12]. In this process, $(\pm, \theta)$ components of the Einstein equations, that can not be captured in the resulting 2-dimensional action, reduces to a condition

$$F_{+-} F_{\theta\pm} = 0. \tag{3}$$

To get static solutions, we assume all the physical variables in our consideration depends only on a space-like variable $x = x^+ + x^-$. This in particular implies $F_{+\theta} = F_{-\theta}$ or, in other words, the two form curvature F of the gauge field consists of the purely electric field F_{-+} and the purely magnetic field $F_{\pm\theta}$. Eq. (3) therefore shows the solutions of our problem have either electric charge or magnetic charge, but not both at the same time. For our further

consideration, we introduce a field A that, in electrically charged case, is defined to satisfy $F_{-+} = dA/dx$, and $F_{\pm\theta} = dA/dx$ in magnetically charged case. Explicitly, the action for the electrically charged sector can be written as

$$I_e = -\int d\bar{x} e^{\rho - 2\phi + \frac{b}{2}f} \left[2\phi'\rho' + \frac{1}{4}f'^2 - \frac{1}{4}e^{\chi f - 2\rho}A'^2 + \frac{\Lambda}{4} \right], \tag{4}$$

while we have the following action for the magnetically charged sector.

$$I_m = -\int d\bar{x} e^{\rho - 2\phi + \frac{b}{2}f} \left[2\phi'\rho' + \frac{1}{4}f'^2 + \frac{1}{4}e^{\chi f + 4\phi}A'^2 + \frac{\Lambda}{4} \right]. \tag{5}$$

Here we introduce a new spatial coordinate $\bar{x}$ via $d\bar{x} = \exp(\rho + bf/2)dx$ and the prime denotes the differentiation with respect $\bar{x}$. We should also impose the static version of the gauge constraints for each case that results from our choice of a conformal gauge.

The underlying symmetry of the theories in our consideration is most apparent when we introduce a set of field redefinitions to new fields X, Y and u given by

$$\begin{pmatrix} X \\ Y \\ u \end{pmatrix} = T \begin{pmatrix} \rho \\ \phi \\ f \end{pmatrix} \tag{6}$$

where the matrix T is

$$T = \begin{pmatrix} \frac{b}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \epsilon \frac{2-b^2}{\sqrt{2|4-b^2|}} & \epsilon \frac{2\sqrt{2}}{\sqrt{|4-b^2|}} & -\epsilon \frac{b}{\sqrt{2|4-b^2|}} \\ 1 & -2 & \frac{b}{2} \end{pmatrix}.$$

We define $\epsilon \equiv (4 - b^2)/|4 - b^2|$, which becomes $+1$ for $|b| < 2$ and -1 for $|b| > 2$. The determinant of the matrix T is $-|b^2 - 4|^{1/2}$. Therefore as long as the parameter b^2 is not 4, the field redefinitions, Eq. (6), are well-defined. In what follows, we find it convenient to use a vector notation $\vec{X} = (X, Y)$ on the 2-dimensional space of fields X and Y with an inner product $\vec{X} \cdot \vec{X} = X^2 + \epsilon Y^2$. The action for both the electric sector and the magnetic sector, then, becomes

$$I = -\int d\bar{x} e^u \left(\frac{1}{2}\vec{X}' \cdot \vec{X}' - \frac{u'^2}{4 - b^2} + \frac{p}{4}\exp(\vec{d} \cdot \vec{X} - \frac{2\chi b + 4}{4 - b^2}u)A'^2 + \frac{\Lambda}{4} \right), \tag{7}$$

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while the gauge constraints can be written as

$$\frac{\vec{X}' \cdot \vec{X}'}{2} - \frac{u'^2}{4 - b^2} + \frac{p}{4} \exp(\vec{d} \cdot \vec{X} - \frac{2\chi b + 4}{4 - b^2} u) A'^2 - \frac{\Lambda}{4} = 0. \tag{8}$$

upon using the static equations of motion. The only difference between the action for the electrically charged sector and the magnetically charged sector is the choice of a 2-dimensional vector $\vec{d}$ and a parameter $p = \pm 1$. For the electrically charged sector, we have the vector $\vec{d} = \vec{d}_e$ where

$$\vec{d}_e = \sqrt{2}(\chi, -\epsilon \frac{2 + \chi b}{\sqrt{|4 - b^2|}}) \tag{9}$$

along with $p = -1$. For the magnetically charged sector, we have $\vec{d} = \vec{d}_m$ where

$$\vec{d}_m = \sqrt{2}(b + \chi, \epsilon \frac{2 - b^2 - \chi b}{\sqrt{|4 - b^2|}}) \tag{10}$$

along with $p = +1$. We notice that the action (7) has a symmetry under the transformation $\bar{x} \to \bar{x} + \alpha$ for an arbitrary constant α. The role of the gauge constraint (8) is to set the Noether charge of this symmetry to zero [12].

Eq. (7) manifestly shows the $O(2)$ symmetry for $|b| < 2$ ($\vec{X} \cdot \vec{X} = X^2 + Y^2$) and $O(1, 1)$ symmetry for $|b| > 2$ ($\vec{X} \cdot \vec{X} = X^2 - Y^2$) when we set $A = 0$, i.e., when there is no $U(1)$ charge. (When $\Lambda = 0$, we have an enhanced symmetry. Then, by a redefinition of $\bar{x}$ and a rescaling of u, $O(2, 1)$ symmetry becomes manifest. We also note the $O(2)$ symmetry was first observed in [4].) These symmetries correspond to the rotation in (X, Y) space. Furthermore, together with the two translational symmetries of X and Y, which are also present when $A = 0$, they constitute the 2-dimensional Euclidean Poincare group or the 2-dimensional Minkowskian Poincare group, depending on the value of b. When we introduce an electric charge, we end up choosing a particular vector $\vec{d}_e$ in the 2-dimensional (X, Y) space. This breaks the Poincare invariance to the Z_2 symmetry, that corresponds to the reflection about the electric axis $\vec{d}_e$, and a single translational symmetry of the fields (X, Y) along the direction perpendicular to the vector $\vec{d}_e$. A similar story holds when we introduce

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a magnetic charge; we break the uncharged Poincare invariance to the Z_2 symmetry about the magnetic axis $\vec{d}_m$ plus one translational symmetry.

We concentrate on the case $0 \leq b < 2$, which contains two prime examples of our concern, namely, the BTZ model ($b = \chi = 0$) and the target space effective action of the string theory ($b = -\chi = \sqrt{2}$). For uncharged cases, we then have the $O(2)$ symmetry. A key observation that shows the existence of an analog of the electric-magnetic duality is that the equality

$$\vec{d}_e \cdot \vec{d}_e = \vec{d}_m \cdot \vec{d}_m = 8\frac{1 + b\chi + \chi^2}{4 - b^2} \tag{11}$$

holds. Therefore, by the reflection about the axis that bisects the electric axis and the magnetic axis, we can transform the electric action exactly into the magnetic action and vice versa. Of course, just as in the case of the electric-magnetic duality of the 4-dimensional Minkowskian space-time, we need to flip the sign of the A'^2 term along with the reflection. This Z_2 transformation, which is also a subgroup of the uncharged $O(2)$ group, is an analog of the electric-magnetic duality in case of the (2+1)-dimensional gravity theories.

An important application of our results is to obtain magnetically charged solutions from the known electrically charged solutions. For this purpose, it is convenient to work out the representation of the above transformations that acts on the space of (ρ, ϕ, f) fields. Using the matrix T and the above considerations, we find

$$\begin{pmatrix} \rho_m \\ \phi_m \\ f_m \end{pmatrix} = \tau_{me} \begin{pmatrix} \rho_e \\ \phi_e \\ f_e \end{pmatrix}, \quad \begin{pmatrix} \rho_e \\ \phi_e \\ f_e \end{pmatrix} = \tau_{em} \begin{pmatrix} \rho_e \\ \phi_e \\ f_e \end{pmatrix}$$

$$\begin{pmatrix} \rho_e \\ \phi_e \\ f_e \end{pmatrix} = \tau_{ee} \begin{pmatrix} \rho_e \\ \phi_e \\ f_e \end{pmatrix}, \quad \begin{pmatrix} \rho_m \\ \phi_m \\ f_m \end{pmatrix} = \tau_{mm} \begin{pmatrix} \rho_m \\ \phi_m \\ f_m \end{pmatrix} \tag{12}$$

where the matrices are given by

$$\tau_{me} = \frac{1}{1 + b\chi + \chi^2} \begin{pmatrix} (b+\chi)^2 & -2 & b+\chi \\ -\frac{1}{2} & \chi^2 & \frac{\chi}{2} \\ -2(b+\chi) & -4\chi & -1 + b\chi + \chi^2 \end{pmatrix}$$

$$\tau_{em} = \frac{1}{1+b\chi+\chi^2} \begin{pmatrix} \chi^2 & -2 & -\chi \\ -\frac{1}{2}(b+\chi)^2 & & -\frac{b+\chi}{2} \\ 2\chi & 4(b+\chi) & -1+b\chi+\chi^2 \end{pmatrix}$$

$$\tau_{ee} = \frac{1}{1+b\chi+\chi^2} \begin{pmatrix} 1 & -2(b+\chi)^2 & b+\chi \\ -\frac{\chi^2}{2} & 1 & \frac{\chi}{2} \\ 2\chi & 4(b+\chi) & -1+b\chi+\chi^2 \end{pmatrix}$$

$$\tau_{mm} = \frac{1}{1+b\chi+\chi^2} \begin{pmatrix} 1 & -2\chi^2 & -\chi \\ -\frac{(b+\chi)^2}{2} & 1 & -\frac{b+\chi}{2} \\ -2(b+\chi) & -4\chi & -1+b\chi+\chi^2 \end{pmatrix}$$

The subscripts e and m for each field represent the solutions in the electrically charged sector and in the magnetically charged sector, respectively. We can straightforwardly verify that τ_{em} transforms Eq. (5) into Eq. (4), τ_{me} transforms Eq. (4) into Eq. (5), τ_{ee} leaves Eq. (4) invariant, and τ_{mm} leaves Eq. (5) invariant. Since τ_{ee} (τ_{mm}) denotes the self-duality within the electrically charged sector (magnetically charged sector), they satisfy $\tau_{ee}^2 = \tau_{mm}^2 = 1$ and $\det \tau_{ee} = \det \tau_{mm} = -1$. We can also straightforwardly verify that $\tau_{me}\tau_{em} = \tau_{em}\tau_{me} = 1$ and $\det \tau_{me} = \det \tau_{em} = 1$. Another interesting observation is the simultaneous transformation $b+\chi \to -\chi$ and $\chi \to -(b+\chi)$ exchanges m and e subscripts. Using Eq.(12), it is straightforward to obtain magnetically charged solutions from the known electrically charged solutions. From [8], some exact electrically charged solutions for the $b = -\chi$ case are available. We recast them in the conformal gauge, apply τ_{me}, and go back to the original gauge, to obtain the following exact magnetically charged solutions. The metric is computed to be

$$ds^2 = (\frac{r^2}{l^2} - M)dt^2 - r^2(\frac{r^2}{l^2} - M)^{b^2/2-1} \left[r^2 + N(\frac{r^2}{l^2} - M)^{b^2/4} - N \right]^{-1} dr^2$$
$$- \left[r^2 + N(\frac{r^2}{l^2} - M)^{b^2/4} - N \right] d\theta^2 \tag{13}$$

and the dilaton field f turns out to be

$$e^{bf} = k^{b^2/2}e^{-bf_1}(\frac{r^2}{l^2} - M)^{-b^2/2} \tag{14}$$

where l, M, k, N and f_1 are constants. In case of the BTZ theory, $b = \chi = 0$, we take the limit $b \to 0$ of Eqs. (13) and (14) setting $M \to M_0$, $l \to l_0$ and $N \to 4Q_M^2/b^2$. The dilaton field in this case becomes a constant $f = -f_1$ and the metric becomes

$$ds^2 = (\frac{r^2}{l_0^2} - M_0)dt^2 - \frac{r^2}{(\frac{r^2}{l_0^2} - M_0)(r^2 + Q_M^2 \ln |\frac{r^2}{l_0^2} - M_0|)}dr^2 - (r^2 + Q_M^2 \ln |\frac{r^2}{l_0^2} - M_0|)d\theta^2,$$

which is exactly the same as the solutions of [9]. For other cases including the most important string effective theory, $b = -\chi = \sqrt{2}$, we obtain new non-trivial magnetically charged solutions. It is interesting that the drastically different solutions in each sector are related by an analog of the usual electric-magnetic duality unlike the (3+1)-dimensional case. In fact, when $\Lambda = 0$, it was found in [2] that the magnetically charged solutions in [13] is related to the electrically charged solutions in [14] via an analog of the electric-magnetic duality. Actually, we can also generate, for example, the dual of the known electrically charged solutions via τ_{ee}. We plan to address the detailed study of the new solutions and the relationships between solutions related by the dual transformations in a future publication.

We can understand more of the structure of the transformations we find so far by noting the relation

$$\cos\alpha = \frac{\vec{d}_m \cdot \vec{d}_e}{\vec{d}_m \cdot \vec{d}_m} = \frac{\chi^2 + b\chi + b^2/2 - 1}{1 + b\chi + \chi^2} \tag{15}$$

that shows the angle α between the electric and the magnetic vector for a given set of parameters (b, χ). By varying the value of α from zero to π, we form a disjoint collection of paths in (b, χ) space. One distinctive case is when a path is represented by $b + 2\chi = 0$. We then have $\cos\alpha = -1$, which means the $\vec{d}_e$ points to the opposite direction to the vector $\vec{d}_m$. The BTZ theory with $b = \chi = 0$ belongs to this case. Clearly, the τ_{me} matrix satisfy $\tau_{me}^2 = 1$, thereby $\tau_{me} = \tau_{em}$, since the two rotations, each of them by 180 degrees, combine to become the identity operation. We also have $\tau_{mm} = \tau_{ee}$, since they represent the reflection

about the electric and magnetic axis, each of which points to the opposite direction to each other. The matrices τ_{me} and τ_{ee}, that commute with each other in this case, generate the direct sum of two Z_2 groups, the Klein four group. In general, when $\alpha = 2\pi/n$ for an integer $n \geq 2$, the smallest positive integer m that satisfies $\tau_{me}^m = 1$ is n. Thus, together with τ_{ee} and τ_{mm}, τ_{me} generates the D_n^*, the symmetry group of n-gon. The most interesting case is when $n = 4$ that gives a situation where the vector $\vec{d}_e$ is perpendicular to the vector $\vec{d}_m$. In this case, $\alpha = \pi/2$, thus

$$\chi^2 + b\chi + b^2/2 - 1 = 0 \tag{16}$$

and the 8-element nonabelian group D_4^* is isomorphic to $O(2, Z)$. The low energy string effective theory with $b = -\chi = \sqrt{2}$ satisfies Eq.(16). Since $O(2, Z)$ is both a natural subgroup of $O(2, 2, Z)$ and $O(2)$, the transformations we find may be considered as a natural subgroup of the T-duality from the string theory. For other values of n, including $n = 2$ for the BTZ theory, τ_{em}, τ_{ee} and τ_{mm} generate a group different from $O(2, Z)$. When combined with $b = -\chi$, which is clear from the fact that the cosmological constant term results from the closed string sector and the $U(1)$ field, from the tree level open string sector, Eq.(16) uniquely determines $b = -\chi = \pm\sqrt{2}$. Both sets of the parameters correctly give the kinetic term of the dilaton field f for the low energy effective string theory after a transformation $f \to \pm f$ in Eq.(1). This shows an interesting possibility; the parameters (for example, b and χ in our case) appearing in the low energy string effective action may be determined to some extent by the requirement of preservation of the T-duality. A similar analysis in higher dimensional cases to the one given here may further verify this idea and the work in this regard is in progress.

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